Maximal Regularity and PDEs

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Chapter 1

Motivation and informal discussion

Consider a family of hypersurfaces $\{\Gamma(t)\}_{t\geq 0}\subset\mathbb{R}^{n+1}$ given as a graph of a height function h over $\mathbb{R}^n$. To be precise we have

$$\Gamma(t)=\{(x,x_{n+1})\in\mathbb{R}^n\times\mathbb{R}:x_{n+1}=h(t,x)\}.$$

Assume furthermore that the evolution of $\Gamma(t)$ is governed by the *surface diffusion law*

$$V_{\Gamma(t)}=-\Delta_{\Gamma(t)}H_{\Gamma(t)},\ t\geq 0,\tag{1.1}$$

where $V_{\Gamma(t)}$ is the normal velocity of $\Gamma(t)$, $H_{\Gamma(t)}=-\mathrm{div}_{\Gamma(t)}\nu_{\Gamma(t)}$ is the mean curvature of $\Gamma(t)$ and the operators $\mathrm{div}_{\Gamma(t)}$ and $\Delta_{\Gamma(t)}$ denote the surface divergence and the Laplace-Beltrami operator, respectively, acting on $\Gamma(t)$. Let us make the convention that the unit normal field $\nu_{\Gamma(t)}$ on $\Gamma(t)$ points from

$$\{(x,x_{n+1})\in\mathbb{R}^n\times\mathbb{R}:x_{n+1}<h(t,x)\}$$

to

$$\{(x,x_{n+1})\in\mathbb{R}^n\times\mathbb{R}:x_{n+1}>h(t,x)\}.$$

It is convenient to rewrite (1.1) in terms of the height function h. To this end let $\beta:=1/\sqrt{1+|\nabla h|^2}$ and denote by δ^{ij} the Kronecker delta. Then we obtain from Prüss & Simonett [22] that

$$\Delta_\Gamma\varphi=(\delta^{kl}-\beta^2\partial_k h\partial_l h)(\partial_k\partial_l\varphi-\beta^2\partial_k\partial_l h\,\partial_m h\,\partial_m\varphi),$$

for functions φ which are smooth enough and

$$H_\Gamma=(\delta^{ij}-\beta^2\partial_i h\partial_j h)\beta\partial_i\partial_j h,$$

where we employed sum convention. Furthermore we have

$$\nu_\Gamma=\beta(-\nabla h,1)^{\mathsf{T}}\quad\text{and}\quad V_\Gamma=\partial_t h(e_{n+1}|\nu_\Gamma)=\beta\partial_t h.$$

Inserting the above expressions into (1.1) yields the equation

$$\partial_t h+\sum_{i,j,k,l=1}^{n}(\delta^{kl}-\beta^2\partial_k h\partial_l h)(\delta^{ij}-\beta^2\partial_i h\partial_j h)\partial_i\partial_j\partial_k\partial_l h=G(\nabla h,\nabla^2 h,\nabla^3 h),\tag{1.2}$$

$$G(\nabla h,\nabla^2 h,\nabla^3 h)=\sum_{|\sigma|=3,|\tau|=2}b_{\sigma\tau}(\nabla h)D^\sigma hD^\tau h+\sum_{|\sigma|=|\tau|=|\chi|=2}c_{\sigma\tau\chi}(\nabla h)D^\sigma hD^\tau hD^\chi h,$$

where $D^\alpha:=(-i)^{|\alpha|}\partial_1^{\alpha_1}\cdots\partial_n^{\alpha_n}$ and $\alpha\in\mathbb{N}_0^n$ is a multiindex.

We note that (1.2) can be written as an abstract quasilinear evolution equation

$$\dot h(t)+A(h(t))h(t)=F(h(t)),\quad t>0,\quad h(0)=h_0,\tag{1.3}$$

in some Banach space X_0, where $A:X\to\mathcal{B}(X_1,X_0)$ is defined by

$$A(h)u:=\sum_{i,j,k,l=1}^{n}(\delta^{kl}-\beta^2\partial_k h\partial_l h)(\delta^{ij}-\beta^2\partial_i h\partial_j h)\partial_i\partial_j\partial_k\partial_l u,$$

with $X_1 \subset X_0$ being the domain of $A(h)$ in X_0 and X is some appropriate space such that

$$X_1 \hookrightarrow X \hookrightarrow X_0,$$

reflecting the fact that the coefficients of the operator $A(h)$ as well as of $F : X \to X_0$ given by

$$F(h) := G(\nabla h, \nabla^2 h, \nabla^3 h)$$

are of lower order.
For solving (1.3), the following strategy may be applied.

1. We write
$$\dot{h}(t) + A(0)h(t) = F(h(t)) + [A(0) - A(h(t))]h(t) =: \tilde{F}(h(t))$$
and define $A_0 : X_1 \to X_0$ by $A_0 := A(0)$.

2. We prove that the linear problem
$$\dot{h}(t) + A_0 h(t) = f, \quad t > 0, \quad h(0) = h_0, \tag{1.4}$$
has for a suitable space $\mathbb{F}$ and for each $(f, h_0) \in \mathbb{F} \times X$ a unique solution $u \in \mathbb{E}$ such that $\mathbb{E} \hookrightarrow \mathbb{F}$.
Hence, the operator $\mathbb{L} : \mathbb{E} \to \mathbb{F} \times X$, defined by
$$\mathbb{L}h := (\dot{h} + A_0 h, h(0))$$
is an isomorphism (maximal regularity comes into play).

3. We prove that the mapping
$$\mathbb{E} \ni h \mapsto \mathbb{L}^{-1}(\tilde{F}(h), h_0) \in \mathbb{E}$$
has a unique fixed point (e.g. by the contraction mapping principle).
Evidently, this fixed point is the unique solution of (1.3).

There are several functional analytic settings for solving (1.4), e.g.

- maximal Hölder regularity (in singular Hölder spaces);
- maximal continuous regularity;
- maximal Sobolev regularity.

We refer to the book of Amann [2] for a detailed exposition of these functional analytic settings. In these lectures, we will only consider maximal Sobolev regularity, also called **maximal L_p-regularity** and we follow Kunstmann & Weis [15] as well as Prüss & Simonett [22].

For the surface diffusion flow, we have $A_0 h = \Delta^2 h$, hence A_0 is the bi-Laplacian, an operator of order 4. One could e.g. choose $X_0 = L_q(\mathbb{R}^n)$, $1 < q < \infty$, so that the domain of A_0 in X_0 is $X_1 = W_q^4(\mathbb{R}^n)$. We are thus looking for solutions

$$h \in W_p^1(\mathbb{R}_+; L_q(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+; W_q^4(\mathbb{R}^n)) =: \mathbb{E},$$

of the linear equation (1.4), where $f \in L_p(\mathbb{R}_+; L_q(\mathbb{R}^n)) =: \mathbb{F}$. The intermediate space X will be derived in Section 4.2.

Chapter 2

Maximal Regularity

2.1 Definitions

Let X_0, X_1 be Banach spaces such that $X_1 \hookrightarrow X_0$ and X_1 is dense in X_0. Suppose that $A : X_1 \to X_0$ is a linear and closed operator ($X_1 = D(A)$ is the domain of A in X_0). We consider the abstract evolution equation

$$\dot{u}(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = 0. \tag{2.1}$$

and have the following

Definition 2.1.1. *Let $1 < p < \infty$ and $J = (0,T)$, $T \in (0,\infty]$. The operator A **has the property of maximal L_p-regularity in** X_0 if for each $f \in L_p(J; X_0)$ there exists a unique solution*

$$u \in W_p^1(J; X_0) \cap L_p(J; X_1) =: \mathbb{E}_1(J)$$

of (2.1)*. If this is the case, we write for short $A \in \mathcal{MR}_p(J, X_0)$ and $\mathcal{MR}_p(\mathbb{R}_+, X_0) =: \mathcal{MR}_p(X_0)$.*

Corollary 2.1.2. *Let $A \in \mathcal{MR}_p(J, X_0)$ for some $p \in (1,\infty)$. Then there exists a constant $C > 0$ such that the unique solution u of* (2.1) *satisfies the estimate*

$$\|u\|_{\mathbb{E}_1(J)} \leq C\|f\|_{L_p(J;X_0)},$$

where

$$\|u\|_{\mathbb{E}_1(J)} := \|\dot{u}\|_{L_p(J;X_0)} + \|Au\|_{L_p(J;X_0)} + \|u\|_{L_p(J;X_0)}.$$

Proposition 2.1.3 ([10, 22])**.** *Let $A \in \mathcal{MR}_p(J, X_0)$ for some $p \in (1,\infty)$.*

1. *If $|J| < \infty$, then $\exists \omega > 0,\ M \geq 1$ such that*

   $$\{\mathrm{Re}\lambda \geq \omega\} \subset \rho(-A) \quad \textit{and} \quad \|\lambda(\lambda + A)^{-1}\|_{\mathcal{B}(X_0)} \leq M,\ \mathrm{Re}\lambda \geq \omega.$$

   In particular, $\omega + A$ is sectorial[1] *with spectral angle $< \pi/2$.*

2. *If $J = \mathbb{R}_+$, then $\exists\ M \geq 1$ such that*

   $$\{\mathrm{Re}\lambda \geq 0\} \subset \rho(-A) \quad \textit{and} \quad \|(1 + |\lambda|)(\lambda + A)^{-1}\|_{\mathcal{B}(X_0)} \leq M,\ \mathrm{Re}\lambda \geq 0.$$

   In particular A is sectorial with spectral angle $< \pi/2$ and

   $$s(-A) := \sup\{\mathrm{Re}\lambda \mid \lambda \in \sigma(-A)\} < 0.$$

Note that by a Theorem of Hille, Proposition 2.1.3 states that if $A \in \mathcal{MR}_p(J, X_0)$, then $-A$ is the generator of an analytic semigroup in X_0 and in case $J = \mathbb{R}_+$ the semigroup is in addition exponentially stable.

[1] $A : D(A) \to X_0$ is called **sectorial**, if (i) A is injective & densely defined with dense range and (ii) $(0,\infty) \subset \rho(-A)$ & $\exists M > 0\ \forall \lambda > 0 : \|\lambda(\lambda + A)^{-1}\|_{\mathcal{B}(X)} \leq M$.
Taylor expansion yields that $\exists \phi \in (0,\pi),\ C_\phi > 0 : \Sigma_{\pi-\phi} \subset \rho(-A)$ and $\forall \lambda \in \Sigma_{\pi-\phi} : \|\lambda(\lambda + A)^{-1}\|_{\mathcal{B}(X_0)} \leq C_\phi$.
The angle $\phi_A := \inf\{\phi \in (0,\pi) : \forall \lambda \in \Sigma_{\pi-\phi} : \|\lambda(\lambda + A)^{-1}\|_{\mathcal{B}(X_0)} \leq C_\phi\}$ is called **spectral angle** of A.

Remark 2.1.4.

1. *If $A \in \mathcal{MR}_p(X_0)$, then $A \in \mathcal{MR}_p(J, X_0)$ for any interval $J = (0, T)$.*

2. *If $|J| < \infty$, then*

$$\forall \omega \in \mathbb{R} : \omega + A \in \mathcal{MR}_p(J, X_0) \Longleftrightarrow A \in \mathcal{MR}_p(J, X_0).$$

3. *If $A \in \mathcal{MR}_p(J, X_0)$ and $s(-A) < 0$, then $A \in \mathcal{MR}_p(X_0)$ (Dore [10]).*

4. *If $A \in \mathcal{MR}_p(J, X_0)$, then $\exists \omega_0 > 0\ \forall \omega > \omega_0 : \omega + A \in \mathcal{MR}_p(X_0)$.*

5. *If $A \in \mathcal{MR}_p(J, X_0)$ for* some *$p \in (1, \infty)$, then $A \in \mathcal{MR}_q(J, X_0)$ for* all *$q \in (1, \infty)$ (Sobolevskii [25]).*

2.2 Conditions for maximal regularity

By Proposition 2.1.3, $A \in \mathcal{MR}_p(X_0)$ implies that A is a sectorial operator in X_0 with spectral angle less than $\pi/2$ and $s(-A) < 0$. Unfortunately, the converse of Proposition 2.1.3 is in general not true as a result of Kalton & Lancien [13] from 2000 shows. For the particular case $X_0 = L_q(\mathbb{R}^n)$, this result reads as follows.

If each generator of an analytic semigroup in $L_q(\mathbb{R}^n)$ has the maximal L_p-regularity property, then $q = 2$.

To prove $A \in \mathcal{MR}_p(X_0)$ for a given operator A is in general a formidable task. To see the difficulties, let A be sectorial with spectral angle $< \pi/2$ and e^{-At} the exponentially stable analytic semigroup generated by $-A$. Define

$$u(t) := \int_0^t e^{-A(t-s)} f(s) ds$$

Then, $A \in \mathcal{MR}_p(X_0)$ provided

$$f \mapsto A \int_0^t e^{-A(t-s)} f(s) ds$$

is bounded from $L_p(\mathbb{R}_+; X_0)$ to $L_p(\mathbb{R}_+; X_0)$. This however is in general nontrivial, since by analyticity of e^{-At},

$$\|Ae^{-At}x\|_{X_0} \leq M \frac{e^{-\omega t}}{t} \|x\|_{X_0}$$

and so there is a non-integrable singularity at $t = 0$.

A first positive result concerning maximal L_p-regularity we want to mention, is

Theorem 2.2.1 (Da Prato & Grisvard [7]). *Let X_0 be a Banach space and A a sectorial invertible operator in X with spectral angle $< \pi/2$ and domain X_1. Then $A \in \mathcal{MR}_p((X_0, X_1)_{\alpha,p})$, where $(X_0, X_1)_{\alpha,p}$ is a real interpolation space with $\alpha \in (0, 1)$ and $p \in (1, \infty)$.*

Example. *Let $X_0 = L_q(\mathbb{R}^n)$, $1 < q < \infty$ and $A = -\Delta_x$ the Laplacian in X_0 with domain $X_1 := W_q^2(\mathbb{R}^n)$. Then $A : X_1 \to X_0$ is closed, densely defined and sectorial in X_0 with spectral angle 0 (see e.g. Abels [1] or Lunardi [18]). Moreover, for any $\omega > 0$, the operator $\omega + A : X_1 \to X_0$ is invertible. It follows from Theorem 2.2.1 that $\omega + A \in \mathcal{MR}_p((X_0, X_1)_{\alpha,p})$ for each $\omega > 0$, where*

$$(X_0, X_1)_{\alpha,p} = (L_q(\mathbb{R}^n), W_q^2(\mathbb{R}^n))_{\alpha,p} = B_{qp}^{2\alpha}(\mathbb{R}^n)$$

is a Besov space (see e.g. [3, 26, 27]). Hence, for any $f \in L_p(\mathbb{R}_+; B_{qp}^{2\alpha}(\mathbb{R}^n))$ there exists a unique solution

$$u \in W_p^1(\mathbb{R}_+; B_{qp}^{2\alpha}(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+; B_{qp}^{2\alpha+2}(\mathbb{R}^n))$$

of the PDE

$$\partial_t u(t, x) + (\omega - \Delta_x) u(t, x) = f(t, x), \quad t > 0,\ x \in \mathbb{R}^n, \quad u(0, x) = 0.$$

We note that not every Banach space can be written as a real interpolation space, e.g. $L_q(\Omega)$, $\Omega \subset \mathbb{R}^n$. To exploit the general strategy, let us write $u(t)$ from above as

$$Au(t) = \int_{\mathbb{R}} k(t-s)\tilde{f}(s)ds = k * \tilde{f}(t), \quad t \in \mathbb{R},$$

where $\tilde{f}(s) := \chi_{(0,\infty)}(s)f(s)$ and $k(s) := \chi_{(0,\infty)}(s)Ae^{-As}$. Applying the Fourier Transformation $\mathcal{F}$ w.r.t. t yields

$$\mathcal{F}(Au) = \mathcal{F}k\mathcal{F}\tilde{f},$$

with

$$(\mathcal{F}k)(\tau) = A\int_0^\infty e^{-i\tau s}e^{-As}ds = A(i\tau + A)^{-1},$$

and hence

$$Au = \mathcal{F}^{-1}A(i\tau + A)^{-1}\mathcal{F}\tilde{f}.$$

Therefore, $Au \in L_p(\mathbb{R}_+; X_0)$ follows if

$$\tau \mapsto A(i\tau + A)^{-1} \in \mathcal{B}(X_0)$$

is a Fourier multiplier in $L_p(\mathbb{R}; X_0)$. Let us collect some classical facts:

- If $X_0 = \mathbb{C}$, then the classical multiplier theorem of Mikhlin (in the 1D-case) states that

$$\left.\begin{aligned} &m \in C^1(\mathbb{R}\setminus\{0\};\mathbb{C}) \\ &\|m\|_M := \max_{\alpha\in\{0,1\}} \sup_{\xi\in\mathbb{R}\setminus\{0\}} |\xi|^\alpha |m^{(\alpha)}(\xi)| < \infty \end{aligned}\right\} \Longrightarrow \mathcal{F}^{-1}m\mathcal{F} \in \mathcal{B}(L_p(\mathbb{R};\mathbb{C})),$$

  and

$$\|\mathcal{F}^{-1}m\mathcal{F}\|_{\mathcal{B}(L_p(\mathbb{R};\mathbb{C}))} \le C\|m\|_M,$$

  see e.g. Abels [1].

- If $X_0 = H$ is a Hilbert space, then a result due to Schwartz [24] (1961) says that

$$\left.\begin{aligned} &m \in C^1(\mathbb{R}\setminus\{0\};\mathcal{B}(H)) \\ &\max_{\alpha\in\{0,1\}} \sup_{\xi\in\mathbb{R}\setminus\{0\}} |\xi|^\alpha \|m^{(\alpha)}(\xi)\|_{\mathcal{B}(H)} < \infty \end{aligned}\right\} \Longrightarrow \mathcal{F}^{-1}m\mathcal{F} \in \mathcal{B}(L_p(\mathbb{R};H)). \tag{2.2}$$

The multiplier result (2.2) yields the following

Theorem 2.2.2 (de Simon [8, 22])**.** *Let $1 < p < \infty$, H be a Hilbert space and A be a sectorial invertible operator in H with spectral angle $< \pi/2$. Then $A \in \mathcal{MR}_p(H)$.*

Example. *Let $H = L_2(\mathbb{R}^n)$, $1 < q < \infty$ and $A = -\Delta_x$ the Laplacian in X_0 with domain $D_A := W_2^2(\mathbb{R}^n) =: H^2(\mathbb{R}^n)$. Then $A : D_A \to H$ is closed, densely defined and sectorial in H with spectral angle 0. Moreover, for any $\omega > 0$, the operator $\omega + A : D_A \to H$ is invertible. It follows from Theorem 2.2.2 that $\omega + A \in \mathcal{MR}_p(H)$ for each $\omega > 0$. Hence, for any $f \in L_p(\mathbb{R}_+; L_2(\mathbb{R}^n))$ there exists a unique solution*

$$u \in W_p^1(\mathbb{R}_+; L_2(\mathbb{R}^n)) \cap L_p(\mathbb{R}_+; H^2(\mathbb{R}^n))$$

of the PDE

$$\partial_t u(t,x) + (\omega - \Delta_x)u(t,x) = f(t,x), \quad t > 0,\ x \in \mathbb{R}^n, \quad u(0,x) = 0.$$

By a result of Lancien, Lancien & Le Merdy [16] (1998), it follows that (2.2) is only true if H is a Hilbert space and therefore the boundedness of $\max_{\alpha\in\{0,1\}} \sup_{\xi\in\mathbb{R}\setminus\{0\}} |\xi|^\alpha \|m^{(\alpha)}(\xi)\|_{\mathcal{B}(X_0)}$ is for non-Hilbert spaces X_0 not sufficient.

At this point, the concept of $\mathcal{R}$-boundedness comes into play (Bourgain [5]).

Definition 2.2.3. *Let X and Y be Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{B}(X,Y)$ is called $\mathcal{R}$**-bounded**, if there is a constant $C > 0$ and $p \in [1,\infty)$, such that for each $N \in \mathbb{N}$, $T_j \in \mathcal{T}$, $x_j \in X$ and for all independent, symmetric, $\{-1,1\}$-valued random variables ε_j on a probability space $(\Omega,\mathcal{M},\mu)$ the inequality*

$$\|\sum_{j=1}^N \varepsilon_j T_j x_j\|_{L_p(\Omega;Y)} \leq C\|\sum_{j=1}^N \varepsilon_j x_j\|_{L_p(\Omega;X)},$$

*is valid. The smallest of such constants $C > 0$ is called $\mathcal{R}$**-bound of** $\mathcal{T}$, which is denoted by $\mathcal{R}(\mathcal{T})$.*

Remark 2.2.4.

- *If $\mathcal{T} \subset \mathcal{B}(X,Y)$ is $\mathcal{R}$-bounded, then $\mathcal{T} \subset \mathcal{B}(X,Y)$ is uniformly bounded (set $N = 1$ in Definition 2.2.3).*
- *If X and Y are Hilbert spaces, then $\mathcal{T} \subset \mathcal{B}(X,Y)$ is $\mathcal{R}$-bounded if and only if $\mathcal{T} \subset \mathcal{B}(X,Y)$ is uniformly bounded. Indeed, let $\mathcal{T} \subset \mathcal{B}(X,Y)$ be uniformly bounded by a constant $C > 0$. Then*

$$\begin{aligned}
\|\sum_{j=1}^N \varepsilon_j T_j x_j\|^2_{L_2(\Omega;Y)} &= \sum_{j,k=1}^N \left[\int_\Omega \varepsilon_j(\omega)\varepsilon_k(\omega)d\mu\right](T_j x_j|T_k x_k)_Y \\
&= \sum_{j=1}^N \left[\int_\Omega \varepsilon_j^2(\omega)d\mu\right]\|T_j x_j\|_Y^2 \\
&\leq C^2\sum_{j=1}^N \left[\int_\Omega \varepsilon_j^2(\omega)d\mu\right]\|x_j\|_X^2 \\
&= C^2\sum_{j,k=1}^N \left[\int_\Omega \varepsilon_j(\omega)\varepsilon_k(\omega)d\mu\right](x_j|x_k)_X = C^2\|\sum_{j=1}^N \varepsilon_j x_j\|^2_{L_2(\Omega;X)}.
\end{aligned}$$

- *Definition 2.2.3 does not depend on $p \in (1,\infty)$. This follows from* Kahanes inequality *(see e.g. Prüss & Simonett [22]).*

Based on the concept of $\mathcal{R}$-boundedness, the following multiplier theorem holds.

Theorem 2.2.5 (Weis [28])**.** *Let X and Y be Banach spaces of class $\mathcal{HT}$ and $p \in (1,\infty)$. Then*

$$\left.\begin{array}{l} m \in C^1(\mathbb{R}\setminus\{0\};\mathcal{B}(X,Y)) \\ \max\limits_{\alpha\in\{0,1\}} \mathcal{R}\left(\{\xi^\alpha m^{(\alpha)}(\xi) \mid \xi \in \mathbb{R}\setminus\{0\}\}\right) < \infty \end{array}\right\} \Longrightarrow \mathcal{F}^{-1}m\mathcal{F} \in \mathcal{B}(L_p(\mathbb{R};X),L_p(\mathbb{R};Y)). \tag{2.3}$$

We note that, by definition, a Banach space X is of class $\mathcal{HT}$ if the Hilbert transform is bounded in $L_p(\mathbb{R};X)$ for some $p \in (1,\infty)$. We list some facts of $\mathcal{HT}$ spaces (see e.g. [2]).

- Every Hilbert space is of class $\mathcal{HT}$.
- Closed subspaces and the dual of $\mathcal{HT}$ spaces are of class $\mathcal{HT}$.
- If X is of class $\mathcal{HT}$, then $L_p(\mathbb{R}^n;X)$ is of class $\mathcal{HT}$ for $p \in (1,\infty)$.
- If $p,q \in (1,\infty)$, $s \in \mathbb{R}$ and $X = \mathbb{R}$, then the scalar versions of the Bessel potential spaces H_p^s, the Besov spaces B_{pq}^s, the Triebel-Lizorkin spaces F_{pq}^s are of class $\mathcal{HT}$.
- Every $\mathcal{HT}$ space is reflexive, hence e.g. L_1 and L_∞ are not of class $\mathcal{HT}$.

The multiplier theorem 2.2.5 yields the following characterization of operators in Banach spaces of class $\mathcal{HT}$ having maximal L_p-regularity.

Theorem 2.2.6 ([15, 22]). *Let X_0 be an Banach space of class $\mathcal{HT}$, $1 < p < \infty$, and let A be a sectorial operator in X_0 with spectral angle $< \pi/2$. Then the following assertions are equivalent.*

1. *For every $f \in L_p(\mathbb{R}_+; X_0)$, there exists a unique function u solving* (2.1) *for a.e. $t > 0$ with $\dot{u}, Au \in L_p(\mathbb{R}_+; X_0)$.*
2. *$\mathcal{R}(\{A(i\tau + A)^{-1} \mid \tau \in \mathbb{R} \setminus \{0\}\}) < \infty$;*
3. *$\mathcal{R}(\{A(\lambda + A)^{-1} \mid \lambda \in \Sigma_\phi\}) < \infty$ for some $\phi > \pi/2$.*

If one of these conditions is satisfied, then $\omega + A \in \mathcal{MR}_p(X_0)$ for each $\omega > 0$, since the analytic semigroup generated by $-(\omega + A)$ in X_0 is exponentially stable, as $s(-(\omega + A)) < 0$. In particular, if in addition $0 \in \rho(A)$, then $A \in \mathcal{MR}_p(X_0)$.

Chapter 3

Application to Partial Differential Equations

3.1 Full space problems

We consider the problem

$$\begin{aligned} \partial_t u + \mathcal{A}(x,D)u &= f \quad \text{in} \quad \mathbb{R}^n, \\ u(0) &= 0 \quad \text{in} \quad \mathbb{R}^n, \end{aligned} \tag{3.1}$$

with the differential operator

$$\mathcal{A}(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x) D^\alpha,$$

where $a_\alpha : \mathbb{R}^n \to \mathbb{C}^{N\times N}$ and $D^\alpha = (-i)^{|\alpha|}\partial_{x_1}^{\alpha_1}\cdots\partial_{x_n}^{\alpha_n}$. The symbol $\mathcal{A}_\#(x,\xi) = \sum_{|\alpha|=2m} a_\alpha(x)\xi^\alpha$ of the principal part

$$\mathcal{A}_\#(x,D) = \sum_{|\alpha|=2m} a_\alpha(x) D^\alpha,$$

of the operator $\mathcal{A}(x,D)$ should satisfy the following condition:

(E) *(Ellipticity*[1] *of the principal part) For all $x \in \mathbb{R}^n \cup \{\infty\}$, $\mathcal{A}_\#(x,\xi)$ is **normally elliptic**, i.e.*

$$\sigma(\mathcal{A}_\#(x,\xi)) \subset \Sigma_\phi, \tag{3.2}$$

for all $\xi \in \mathbb{R}^n$, $|\xi| = 1$, where

$$\phi_{\mathcal{A}} := \inf\{\phi \in [0,\pi) \mid \text{(3.2) } \textit{holds}\} < \pi/2.$$

In $X_0 := L_q(\mathbb{R}^n;\mathbb{C}^N)$ we define operators $A_q, A_{q,\#} : X_1 \to X_0$ with $X_1 := W_q^2(\mathbb{R}^n;\mathbb{C}^N)$ by

$$A_q u := \mathcal{A}(x,D)u \quad A_{q,\#}u := \mathcal{A}_\#(x,D)u,$$

so that A_q and $A_{q,\#}$ are densely defined .

We consider first the case of constant coefficients $a_\alpha(x) = a_\alpha \in \mathbb{C}^{N\times N}$. By the classical Mikhlin multiplier theorem (in the nD-case), there exists a constant $C > 0$ such that

$$\|D^\beta u\|_{X_0} \leq C\|A_{q,\#}u\|_{X_0}, \quad |\beta| = 2m$$

for any $u \in X_1$. Therefore, A_q and $A_{q,\#}$ are closed in X_0.

Theorem 3.1.1. *Let $a_\alpha(x) = a_\alpha \in \mathbb{C}^{N\times N}$ and let assumption **(E)** be satisfied. Then $\overline{\mathbb{C}_+}\setminus\{0\} \subset \rho(-A_{q,\#})$ and*

$$\mathcal{R}(\{\lambda(\lambda + A_{q,\#})^{-1} \mid \lambda \in \overline{\mathbb{C}_+}\setminus\{0\}\}) < \infty.$$

In particular, $\omega + A_{q,\#} \in \mathcal{MR}_p(X_0)$ for any $\omega > 0$, since the analytic semigroup generated by $-(\omega + A_{q,\#})$ in X_0 is exponentially stable, as $s(-(\omega + A_{q,\#})) < 0$.

[1] For a journey through the elliptic jungle, we refer to [22, Chapter 6].

Example. *We consider the negative Laplacian resp. the bi-Laplacian* $\mathcal{A}_{\#}(D) = (-\Delta)^k$*,* $k \in \{1,2\}$*, with symbol*

$$\mathcal{A}_{\#}(\xi) = |\xi|^{2k}.$$

It follows that

$$\sigma(\mathcal{A}_{\#}(\xi)) = \{1\},$$

for all $\xi \in \mathbb{R}^n$ *with* $|\xi| = 1$ *and therefore* $\mathcal{A}_{\#}(\xi)$ *is normally elliptic. By Theorem 3.1.1 and for any* $\omega > 0$*, it holds that*

$$\omega + A_{q,\#} \in \mathcal{MR}_p(X_0),$$

since the analytic semigroup generated by $-(\omega + A_{q,\#})$ *in* X_0 *is exponentially stable, as* $s(-(\omega + A_{q,\#})) < 0$*.*

Since $\mathcal{R}$-bounds behave well under perturbations, Theorem 3.1.1 can be extended to small perturbations

$$\sum_{|\alpha|=2m} b_\alpha(x) D^\alpha, \qquad \sum_{|\alpha|=2m} \|b_\alpha\|_{L_\infty(\mathbb{R}^n)} \leq \varepsilon$$

of the constant coefficients case. A localization procedure finally yields the following

Theorem 3.1.2 ([22])**.** *Let assumption **(E)** be satisfied and assume that*

$$a_\alpha \in C_\ell(\mathbb{R}^n; \mathbb{C}^{N\times N}) \quad (|\alpha| = 2m)$$

$$a_\alpha \in L_\infty(\mathbb{R}^n; \mathbb{C}^{N\times N}) \quad (|\alpha| < 2m).$$

Then $\exists \omega_0 > 0$ *such that* $\Sigma_\phi \subset \rho(-(\omega_0 + A_q))$ *and*

$$\mathcal{R}(\{\lambda(\lambda + \omega_0 + A_q)^{-1} \mid \lambda \in \Sigma_\phi\}) < \infty$$

for some $\phi > \pi/2$*. In particular,* $\omega + \omega_0 + A_q \in \mathcal{MR}_p(X_0)$ *for any* $\omega > 0$ *since the analytic semigroup generated by* $-(\omega + \omega_0 + A_q)$ *in* X_0 *is exponentially stable, as* $s(-(\omega + \omega_0 + A_q)) < 0$*.*

3.2 General domains

In case that $\Omega \subset \mathbb{R}^n$ is open and bounded with boundary $\partial\Omega \in C^{2m}$, we consider the problem

$$\begin{aligned} \partial_t u + \mathcal{A}(x,D)u &= f \quad \text{in} \quad \Omega, \\ \mathcal{B}_j(x,D)u &= g_j \quad \text{on} \quad \partial\Omega,\ j \in \{1,\dots,m\} \\ u(0) &= 0 \quad \text{in} \quad \Omega. \end{aligned} \tag{3.3}$$

with the differential operators

$$\mathcal{A}(x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x) D^\alpha, \quad x \in \Omega$$

and

$$\mathcal{B}_j(x,D) = \sum_{|\beta|\leq m_j} b_{j\beta}(x) D^\beta, \quad m_j < 2m, \quad x \in \partial\Omega,\ j \in \{1,\dots,m\}$$

with coefficients

$$a_\alpha \in C(\overline{\Omega}; \mathbb{C}^{N\times N}) \quad (|\alpha| = 2m)$$

$$a_\alpha \in L_\infty(\Omega; \mathbb{C}^{N\times N}) \quad (|\alpha| < 2m)$$

and

$$b_{j,\beta} \in C^{2m-m_j}(\partial\Omega; \mathbb{C}^{N\times N}) \quad (|\beta| \leq m_j).$$

By a localization procedure, change of coordinates and perturbation arguments, one obtains full-space problems for the charts which do not intersect the boundary $\partial\Omega$ and problems in the half-space

$$\mathbb{R}^n_+ := \{x = (x', y) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid y > 0\}$$

induced by the charts which intersect the boundary $\partial\Omega$. The corresponding problem for the case of constant coefficients for the principle parts reads

$$\begin{aligned} \partial_t u + \mathcal{A}_\#(D)u = f \quad &\text{in} \quad \mathbb{R}^n_+, \\ \mathcal{B}_{j\#}(D)u = g_j \quad &\text{on} \quad \partial\mathbb{R}^n_+,\ j \in \{1,\dots,m\} \\ u(0) = 0 \quad &\text{in} \quad \mathbb{R}^n_+. \end{aligned} \tag{3.4}$$

Note that w.l.o.g. one may always assume $f = 0$ in (3.4), by extending f to $\tilde{f}$ on $\mathbb{R}^n$ by zero and solving a full space problem. Taking Laplace transform w.r.t t and Fourier-Transform w.r.t. $x' \in \mathbb{R}^{n-1}$ we obtain

$$\begin{aligned} \lambda v(y) + \mathcal{A}_\#(\xi', D_y)v(y) = 0 \quad &, \quad y > 0, \\ \mathcal{B}_{j\#}(\xi', D_y)v(0) = g_j \quad &, \quad y = 0,\ j \in \{1,\dots,m\}. \end{aligned} \tag{3.5}$$

This is a boundary value problem for an ordinary differential equation on $\mathbb{R}_+$ of order $2m$ in the variable y with parameters $\lambda \in \mathbb{C}$ and $\xi' \in \mathbb{R}^{n-1}$. The unique solvability of (3.5) is ensured by

(LS) *(Lopatinskii-Shapiro condition) For all $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in \Sigma_\phi$ (for some $\phi > \pi/2$), the problem* (3.5) *has a unique solution $v \in C_0(\mathbb{R}_+; \mathbb{C}^N)$ for any given $g_j \in \mathbb{C}^N$.*

The philosophy of the Lopatinskii-Shapiro condition is to check whether the boundary conditions $\mathcal{B}_j(x, D)$ fit to the operator $\mathcal{A}(x, D)$.

Based on a solution formula for the solution $v(y)$ of (3.5) and some very technical kernel estimates, one obtains an analogous result to Theorem 3.1.1 for the case of the half space with constant coefficients. This in turn can be extended to coefficients having a small perturbation in the principal parts. The localization procedure for the domain Ω then yields the following result.

Theorem 3.2.1 ([22]). *Let $\Omega \subset \mathbb{R}^n$ open and bounded with boundary $\partial\Omega \in C^{2m}$ and let $p, q \in (1, \infty)$. Assume that the ellipticity condition **(E)** for each $x \in \overline{\Omega}$ and the Lopatinskii-Shapiro condition **(LS)** for each $x \in \partial\Omega$ are satisfied.*

Then there exists $\omega_0 > 0$ such that for each $\omega \geq \omega_0$ the problem

$$\begin{aligned} \partial_t u + \omega u + \mathcal{A}(x, D)u = f \quad &\text{in} \quad \Omega, \\ \mathcal{B}_j(x, D)u = 0 \quad &\text{on} \quad \partial\Omega,\ j \in \{1,\dots,m\} \\ u(0) = 0 \quad &\text{in} \quad \Omega, \end{aligned} \tag{3.6}$$

has a unique solution

$$u \in W^1_p(\mathbb{R}_+; L_q(\Omega; \mathbb{C}^N)) \cap L_p(\mathbb{R}_+; W^{2m}_q(\Omega; \mathbb{C}^N))$$

if and only if $f \in L_p(\mathbb{R}_+; L_q(\Omega; \mathbb{C}^N))$. In other words, for the operator $A_q u := \mathcal{A}(x, D)u$ in $X_0 := L_q(\Omega; \mathbb{C}^N)$ with domain

$$X_1 := \{u \in W^{2m}_q(\Omega; \mathbb{C}^N) \mid \mathcal{B}_j(x, D)u = 0,\ x \in \partial\Omega,\ j \in \{1,\dots,m\}\}$$

there holds $\omega + A_q \in \mathcal{MR}_p(X_0)$ for each $\omega \geq \omega_0$.

Example. *We consider the negative Laplacian $\mathcal{A}(x, D) = -\Delta$ with either one of the boundary conditions*

- $\mathcal{B}(x, D) = u|_{\partial\Omega}$ *(Dirichlet BC) or;*
- $\mathcal{B}(x, D) = \nabla u|_{\partial\Omega} \cdot \nu_{\partial\Omega}$ *(Neumann BC);*

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in C^2$ and $\nu_{\partial\Omega}$ is the outer unit normal field on $\partial\Omega$.

We already know that $\mathcal{A}(D)$ is normally elliptic. Thus, it remains to verify the Lopatinskii-Shapiro condition. To this end, we have to consider the parameter-dependend ODE

$$\lambda v(y) + (|\xi'|^2 - \partial_y^2)v(y) = 0, \quad (\lambda, \xi') \in \Sigma_\phi \times \mathbb{R}^{n-1},\ \phi > \pi/2,$$

with $v(0) = g$ in case of Dirichlet BCs or $\partial_y v(0) = -g$ in case of Neumann BCs. We note that the general solution of the ODE is given by

$$v(y) = c_1 e^{-y\sqrt{\lambda + |\xi'|^2}} + c_2 e^{y\sqrt{\lambda + |\xi'|^2}},\ y \geq 0,\ c_j \in \mathbb{C}.$$

Since $v(y) \to 0$ as $y \to \infty$, we necessarily have $c_2 = 0$ hence

$$v(y) = c_1 e^{-y\sqrt{\lambda+|\xi'|^2}} \quad \text{and} \quad \partial_y v(y) = -c_1\sqrt{\lambda+|\xi'|^2}e^{-y\sqrt{\lambda+|\xi'|^2}}.$$

*The initial conditions imply $c_1 = g$ in case of a Dirichlet BC, or $c_1 = g/\sqrt{\lambda+|\xi'|}$ in case of a Neumann BC and therefore, **(LS)** is fulfilled.*

Chapter 4

Trace spaces and time-weights

4.1 Time-weighted spaces

As a generalization of the classical L_p-spaces in Chapter 2, we consider now L_p-spaces with certain time-weights. For an arbitrary Banach space X and for $1 < p < \infty$, we define the weighted L_p-space $L_{p,\mu}(\mathbb{R}_+; X)$ by

$$L_{p,\mu}(\mathbb{R}_+; X) := \{f \in L_{1,loc}(\mathbb{R}_+; X) : [t \mapsto t^{1-\mu} f(t)] \in L_p(\mathbb{R}_+; X)\},$$

where $\mu \in (1/p, 1]$. Note that $L_p(\mathbb{R}_+; X)$ is the classical Bochner-Lebesgue space and evidently, it holds that $L_{p,1}(\mathbb{R}_+; X) = L_p(\mathbb{R}_+; X)$.

The weighted Sobolev-space $W^1_{p,\mu}(\mathbb{R}_+; X)$ is accordingly defined by

$$W^1_{p,\mu}(\mathbb{R}_+; X) := \{u \in L_{p,\mu}(\mathbb{R}_+; X) \cap W^1_{1,loc}(\mathbb{R}_+; X) : \dot{u} \in L_{p,\mu}(\mathbb{R}_+; X)\}.$$

The spaces $L_{p,\mu}(\mathbb{R}_+; X)$ and $W^1_{p,\mu}(\mathbb{R}_+; X)$ are equipped with the norms

$$\|f\|_{L_{p,\mu}(\mathbb{R}_+;X)} := \left(\int_0^\infty \|t^{1-\mu} f(t)\|_X^p \, dt\right)^{1/p}$$

and

$$\|u\|_{W^1_{p,\mu}(\mathbb{R}_+;X)} := \left(\|u\|^p_{L_{p,\mu}(\mathbb{R}_+;X)} + \|\dot{u}\|^p_{L_{p,\mu}(\mathbb{R}_+;X)}\right)^{1/p},$$

respectively, which turn them into Banach spaces. Moreover, it can be shown that

$$W^1_{p,\mu}(\mathbb{R}_+; X) \hookrightarrow W^1_{1,loc}(\overline{\mathbb{R}_+}; X),$$

for any $\mu \in (1/p, 1]$, hence any function $u \in W^1_{p,\mu}(\mathbb{R}_+; X)$ has a well-defined trace $u(0)$ in X.

In accordance with Definition 2.1.1 we have the following

Definition 4.1.1. *Let $1 < p < \infty$ and $\mu \in (1/p, 1]$. The operator A* ***has the property of maximal $L_{p,\mu}$-regularity in*** *X_0 if for each $f \in L_{p,\mu}(\mathbb{R}_+; X_0)$ there exists a unique solution*

$$u \in W^1_{p,\mu}(\mathbb{R}_+; X_0) \cap L_{p,\mu}(\mathbb{R}_+; X_1)$$

of (2.1)*. If this is the case, we write for short $A \in \mathcal{MR}_{p,\mu}(X_0)$ and*

$$\mathcal{MR}_{p,1}(X_0) =: \mathcal{MR}_p(X_0)$$

if $\mu = 1$.

Corollary 4.1.2. *Let $A \in \mathcal{MR}_{p,\mu}(X_0)$ for some $p \in (1, \infty)$. Then there exists a constant $C > 0$ such that the unique solution u of* (2.1) *satisfies the estimate*

$$\|u\|_{\mathbb{E}_{1,\mu}(\mathbb{R}_+)} \le C \|f\|_{L_{p,\mu}(\mathbb{R}_+;X_0)},$$

where

$$\|u\|_{\mathbb{E}_{1,\mu}(\mathbb{R}_+)} := \|\dot{u}\|_{L_{p,\mu}(\mathbb{R}_+;X_0)} + \|Au\|_{L_{p,\mu}(\mathbb{R}_+;X_0)} + \|u\|_{L_{p,\mu}(\mathbb{R}_+;X_0)}.$$

The following important theorem draws a connection between the classes $\mathcal{MR}_p(X_0)$ and $\mathcal{MR}_{p,\mu}(X_0)$.

Theorem 4.1.3 (Prüss & Simonett [21]). *For all $1 < p < \infty$ and $\mu \in (1/p, 1]$ the following assertions are equivalent:*

1. $A \in \mathcal{MR}_{p,\mu}(X_0)$;
2. $A \in \mathcal{MR}_p(X_0)$.

In particular, Theorem 4.1.3 asserts that the case of classical (unweighted) maximal L_p-regularity extrapolates to the weighted maximal $L_{p,\mu}$-regularity setting without any additional assumptions, as long as $\mu \in (1/p, 1]$.

4.2 Trace spaces

So far, we have only considered trivial initial values $u(0) = 0$. Let us consider the question, under which assumptions on u_0, there exists a unique solution

$$u \in W^1_{p,\mu}(\mathbb{R}_+; X_0) \cap L_{p,\mu}(\mathbb{R}_+; X_1)$$

of the problem

$$\dot{u}(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = u_0, \tag{4.1}$$

provided $A \in \mathcal{MR}_p(X_0)$ and $f \in L_{p,\mu}(\mathbb{R}_+; X_0)$. Recall that $A : X_1 \to X_0$ is a closed and densely defined operator in X_0. By Proposition 2.1.3, $-A$ generates an exponentially stable analytic semigroup e^{-At} in X_0. Suppose $[t \mapsto Ae^{-At}x] \in L_{p,\mu}(\mathbb{R}_+; X_0)$. Then, by definition of the weighted L_p-spaces, there holds

$$\int_0^\infty \|t^{1-\mu} Ae^{-At}x\|^p_{X_0} dt < \infty.$$

This motivates the definition of intermediate spaces.

Definition 4.2.1 (Lunardi [18]). *Let A be a sectorial and invertible operator with spectral angle $< \pi/2$, $\alpha \in (0, 1)$ and $p \in [1, \infty)$. We define*

$$D_A(\alpha, p) := \left\{ x \in X_0 \mid [x]_{\alpha,p} := \left(\int_0^\infty \|t^{1-\alpha} Ae^{-At}x\|^p_{X_0} \frac{dt}{t} \right)^{1/p} < \infty \right\}.$$

If $D_A(\alpha, p)$ is equipped with the norm $\|x\|_{\alpha,p} := \|x\|_{X_0} + [x]_{\alpha,p}$, then $D_A(\alpha, p)$ is a Banach space.

There is a connection between the intermediate spaces $D_A(\alpha, p)$ and real interpolation spaces $(X_0, X_1)_{\theta,p}$, $\theta \in (0, 1)$.

Proposition 4.2.2 (Lunardi [18]). *Let A be a sectorial invertible operator in $X0$ with spectral angle $< \pi/2$. Suppose that $\alpha \in (0, 1)$ and $p \in [1, \infty)$. Then*

$$D_A(\alpha, p) = (X_0, X_1)_{\alpha,p}$$

up to equivalent norms, where X_1 is the domain of A in X_0. Furthermore,

$$X_1 \hookrightarrow D_A(\beta, p) \hookrightarrow D_A(\alpha, p) \hookrightarrow X_0, \quad 0 < \alpha < \beta < 1.$$

For $\alpha = \mu - 1/p$, $\mu \in (1/p, 1]$, it follows that

$$x \in (X_0, X_1)_{\mu-1/p,p} \iff x \in D_A(\mu - 1/p, p) \iff \|[t \mapsto t^{1-\mu} Ae^{-At}x]\|_{L_p(\mathbb{R}_+;X_0)} < \infty.$$

Hence, we have the following result

Theorem 4.2.3. *Let $1 < p < \infty$, $\mu \in (1/p, 1]$ and $A \in \mathcal{MR}_p(X_0)$. Then there exists a unique solution*

$$u \in W^1_{p,\mu}(\mathbb{R}_+; X_0) \cap L_{p,\mu}(\mathbb{R}_+; X_1) =: \mathbb{E}_{1,\mu}(\mathbb{R}_+)$$

of

$$\dot{u}(t) + Au(t) = f(t), \quad t > 0, \quad u(0) = u_0, \tag{4.2}$$

if and only if

1. *$f \in L_{p,\mu}(\mathbb{R}_+; X_0)$;*
2. *$u_0 \in (X_0, X_1)_{\mu-1/p,p}$.*

Furthermore, there exists a constant $C > 0$ such that

$$\|u\|_{\mathbb{E}_{1,\mu}(\mathbb{R}_+)} \leq C \left(\|f\|_{L_{p,\mu}(\mathbb{R}_+;X_0)} + \|u_0\|_{(X_0,X_1)_{\mu-1/p,p}} \right).$$

Remark 4.2.4. *Theorem 4.2.3 asserts in particular, that the regularity of the initial value can be reduced by decreasing the exponent $\mu \in (1/p, 1]$ of the time-weight. This in turn implies that the number of compatibility conditions in the context of initial-boundary value problems for parabolic partial differential equations may be reduced to a minimum.*

Example. *We consider the negative Laplacian $A_q u := -\Delta u$ in $X_0 := L_q(\Omega; \mathbb{C}^N)$ with domain*

$$X_1 := \{u \in W_q^2(\Omega; \mathbb{C}^N) \mid u|_{\partial\Omega} = 0 \textit{ on } \partial\Omega\}.$$

Then we already know that $\exists \omega > 0 : \omega + A_q \in \mathcal{MR}_p(X_0)$, see Section 3.2. The trace space $X_{\gamma,\mu} = (X_0, X_1)_{\mu-1/p,p}$ is then computed to the result

$$X_{\gamma,\mu} = \begin{cases} \{u \in B_{qp}^{2\mu-2/p}(\Omega; \mathbb{C}^N) \mid u|_{\partial\Omega} = 0 \textit{ on } \partial\Omega\}, & \textit{if } \mu > 1/p + 1/(2q), \\ B_{qp}^{2\mu-2/p}(\Omega; \mathbb{C}^N), & \textit{if } \mu < 1/p + 1/(2q). \end{cases}$$

Therefore, for each $(f, u_0) \in L_{p,\mu}(\mathbb{R}_+; L_q(\Omega; \mathbb{C}^N)) \times X_{\gamma,\mu}$ there exists a unique solution

$$u \in W_{p,\mu}^1(\mathbb{R}_+; L_q(\Omega; \mathbb{C}^N)) \cap L_{p,\mu}(\mathbb{R}_+; W_q^2(\Omega; \mathbb{C}^N))$$

of the parabolic initial-boundary-value problem

$$\begin{aligned} \partial_t u + \omega u - \Delta u &= f \quad \textit{in} \quad \Omega, \\ u|_{\partial\Omega} &= 0 \quad \textit{on} \quad \partial\Omega, \\ u(0) &= u_0 \quad \textit{in} \quad \Omega. \end{aligned}$$

*In particular, if $1/p < \mu < 1/p + 1/(2q)$, there is **NO** compatibility condition for u_0 on $\partial\Omega$.*

One can even prove a more general result for the trace at $t = 0$.

Lemma 4.2.5 ([22]). *Let $1 < p < \infty$ and $\mu \in (1/p, 1]$. Then the trace operator*

$$\mathrm{tr} : \mathbb{E}_{1,\mu}(\mathbb{R}_+) \to (X_0, X_1)_{\mu-1/p,p} \quad \mathrm{tr}\, u := u(0)$$

is linear, surjective and bounded, hence

$$\exists C > 0\ \forall u \in \mathbb{E}_{1,\mu}(\mathbb{R}_+) : \|\mathrm{tr}\, u\|_{(X_0,X_1)_{\mu-1/p,p}} \leq C\|u\|_{\mathbb{E}_{1,\mu}(\mathbb{R}_+)}.$$

Moreover,

$$\mathbb{E}_{1,\mu}(\mathbb{R}_+) \hookrightarrow BUC(\mathbb{R}_+; (X_0, X_1)_{\mu-1/p,p}).$$

Let us point out another advantage of working in the setting of weighted L_p-spaces. To see the benefit, observe that for all $\tau > 0$, the estimate

$$\tau^{1-\mu}\|u\|_{\mathbb{E}_{1,1}(\tau,\infty)} \leq \|u\|_{\mathbb{E}_{1,\mu}(\tau,\infty)} \leq \|u\|_{\mathbb{E}_{1,\mu}(\mathbb{R}_+)}$$

for the solution u of (4.2) holds, hence

$$u \in W_{p,loc}^1((0,\infty); X_0) \cap L_{p,loc}((0,\infty), X_1) \hookrightarrow C((0,\infty); X_{\gamma,1}),$$

This shows that the solution $u(t)$ of (4.2) with initial value $u_0 \in X_{\gamma,\mu} = (X_0, X_1)_{\mu-1/p,p}$ regularizes instantaneously for $t > 0$ provided $\mu < 1$.

Chapter 5

Quasilinear parabolic evolution equations

5.1 Local well-posedness

Consider the quasilinear evolution equation

$$\dot{u}(t) + A(u(t))u(t) = F(u(t)),\ t > 0, \quad u(0) = u_0, \tag{5.1}$$

under the assumption that there exist two Banach spaces X_0, X_1, with dense embedding $X_1 \hookrightarrow X_0$ such that the nonlinear mappings (A, F) satisfy

$$(A, F) \in C^{1-}(V_\mu; \mathcal{B}(X_1, X_0) \times X_0), \tag{5.2}$$

where $V_\mu \subset (X_0, X_1)_{\mu-1/p,p} =: X_{\gamma,\mu}$ is open and nonempty for some $\mu \in (1/p, 1]$. The main result of this section reads as follows.

Theorem 5.1.1 ([14]). *Let $p \in (1, \infty)$, $u_0 \in V_\mu$ be given and suppose that (A, F) satisfy* (5.2) *for some $\mu \in (1/p, 1]$. Assume in addition that $A(u_0) \in \mathcal{MR}_p(X_0)$. Then there exist $T = T(u_0) > 0$ and $\varepsilon = \varepsilon(u_0) > 0$, such that $\bar{B}_\varepsilon^{X_{\gamma,\mu}}(u_0) \subset V_\mu$ and such that the problem*

$$\dot{u}(t) + A(u(t))u(t) = F(u(t)),\ t > 0, \quad u(0) = u_1,$$

has a unique solution

$$u(\cdot, u_1) \in W^1_{p,\mu}((0,T); X_0) \cap L_{p,\mu}((0,T); X_1) \cap C([0,T]; V_\mu),$$

on $[0, T]$, for any initial value $u_1 \in \bar{B}_\varepsilon^{X_{\gamma,\mu}}(u_0)$. Furthermore there exists a constant $c = c(u_0) > 0$ such that for all $u_1, u_2 \in \bar{B}_\varepsilon^{X_{\gamma,\mu}}(u_0)$ the estimate

$$||u(\cdot, u_1) - u(\cdot, u_2)||_{\mathbb{E}_{1,\mu}(0,T)} \le c\|u_1 - u_2\|_{X_{\gamma,\mu}}$$

is valid.

Remark 5.1.2. *A benefit of Theorem 5.1.1 is that the local existence time $T = T(u_0)$ is locally uniform. Moreover, Theorem 5.1.1 shows that the space $X_{\gamma,\mu} = (X_0, X_1)_{\mu-1/p,p}$ is the natural phase space for the semi-flow $[u_0 \mapsto u(t, u_0)]$ generated by* (5.1).

Example. *Let $X_0 = L_q(\mathbb{R}^n)$, $X_1 = W^4_q(\mathbb{R}^n)$ and hence*

$$X_{\gamma,\mu} = (X_0, X_1)_{\mu-1/p,p} = B^{4\mu-4/p}_{qp}(\mathbb{R}^n).$$

In the sequel, we assume

$$\frac{4}{p} + \frac{n}{q} < 1,$$

so that $X_{\gamma,\mu} \hookrightarrow BC^3(\mathbb{R}^n)$ provided

$$4\mu > 3 + \frac{4}{p} + \frac{n}{q}.$$

We consider the surface diffusion flow (1.1) *in the graph setting, rewritten as*

$$\dot{h}(t) + A(h(t))h(t) = F(h(t)), \quad t > 0, \quad h(0) = h_0, \tag{5.3}$$

where $A : X_\gamma \to \mathcal{B}(X_1, X_0)$ *is given by*

$$A(h)u := \sum_{i,j,k,l=1}^{n} (\delta^{kl} - \beta^2 \partial_k h \partial_l h)(\delta^{ij} - \beta^2 \partial_i h \partial_j h)\partial_i \partial_j \partial_k \partial_l u,$$

for $h \in X_{\gamma,\mu}$, $u \in X_1$ *and* $F : X_{\gamma,\mu} \to X_0$ *is given by* $F(h) = G(\nabla h, \nabla^2 h, \nabla^3 h)$. *Under the above conditions on* p, q, μ, *it follows that* (A, F) *satisfy* (5.2) *by the theory of Nemytskii operators. Moreover,* $A(0)u = \Delta^2 u$ *is the bi-Laplacian in* $X_0 = L_q(\mathbb{R}^n)$ *which is normally elliptic by Section 3.1, hence* $\omega + A(0) \in \mathcal{MR}_p(X_0)$ *for any* $\omega > 0$. *Replacing* $F(h)$ *by* $\tilde{F}(h) := F(h) + \omega h$, *Theorem 5.1.1 yields a local-in-time solution*

$$h(\cdot, h_0) \in W^1_{p,\mu}((0,T); L_q(\mathbb{R}^n)) \cap L_{p,\mu}((0,T); W^4_q(\mathbb{R}^n)) \cap C([0,T]; B^{4\mu-4/p}_{qp}(\mathbb{R}^n))$$

of (5.3) *for any initial value* $h_0 \in B^{4\mu-4/p}_{qp}(\mathbb{R}^n)$ *with* $\|h_0\|_{X_{\gamma,\mu}} \le \varepsilon$.

The next result provides information about the continuation of local solutions.

Corollary 5.1.3. *Let the assumptions of Theorem 5.1.1 be satisfied and assume that* $A(v) \in \mathcal{MR}_p(X_0)$ *for all* $v \in V_\mu$. *Then the solution* $u(t)$ *of* (5.1) *with initial value* $u_0 \in V_\mu$ *has a maximal interval of existence* $J(u_0) = [0, t^+(u_0))$.

The mapping $[u_0 \mapsto t^+(u_0)] : V_\mu \to (0, \infty)$ *is lower-semicontinuous.*

5.2 Relative compactness and global existence

Let $u_0 \in V_\mu$ be given. Suppose that (A, F) satisfy (5.2) and $A(v) \in \mathcal{MR}_p(X_0)$ for all $v \in V_\mu$ and for some $\mu \in (1/p, 1)$, where $J = [0, T]$ or $J = \mathbb{R}_+$. In the sequel we assume that the unique solution of (5.1) satisfies $u \in BC([\tau, t^+(u_0)); V_\mu \cap X_\gamma)$ for some $\tau \in (0, t^+(u_0))$ and

$$\operatorname{dist}(u(t), \partial V_\mu) \ge \eta > 0 \tag{5.4}$$

for all $t \in J(u_0)$. Suppose furthermore that

$$X_\gamma \hookrightarrow\hookrightarrow X_{\gamma,\mu}, \quad \mu \in (1/p, 1). \tag{5.5}$$

It follows from the boundedness of $u(t)$ in X_γ that the set $\{u(t)\}_{t \in J(u_0)} \subset V_\mu$ is relatively compact in $X_{\gamma,\mu}$, provided $\mu \in (1/p, 1)$. By (5.4) it holds that $\mathcal{V} := \overline{\{u(t)\}}_{t \in J(u_0)}$ is a proper subset of V_μ. Applying Theorem 5.1.1 we find for each $v \in \mathcal{V}$ numbers $\varepsilon(v) > 0$ and $\delta(v) > 0$ such that $B^{X_{\gamma,\mu}}_{\varepsilon(v)}(v) \subset V_\mu$ and all solutions of (5.1) which start in $B^{X_{\gamma,\mu}}_{\varepsilon(v)}(v)$ have the common interval of existence $[0, \delta(v)]$. Therefore the set

$$\bigcup_{v \in \mathcal{V}} B^{X_{\gamma,\mu}}_{\varepsilon(v)}(v)$$

is an open covering of $\mathcal{V}$ and by compactness of $\mathcal{V}$ there exist $N \in \mathbb{N}$ and $v_k \in \mathcal{V}$, $k = 1, \dots, N$, such that

$$\mathcal{U} := \bigcup_{k=1}^{N} B^{X_{\gamma,\mu}}_{\varepsilon_k}(v_k) \supset \mathcal{V} = \overline{\{u(t)\}}_{t \in J(u_0)} \supset \{u(t)\}_{t \in J(u_0)},$$

where $\varepsilon_k := \varepsilon(v_k)$, $k = 1, \dots, N$. To each of these balls corresponds an interval of existence $[0, \delta_k]$, $\delta_k > 0$, $k = 1, \dots, N$. Consider the problem

$$\dot{v} + A(v)v = F(v), \ s > 0, \quad v(0) = u(t), \tag{5.6}$$

where $t \in J(u_0)$ is fixed and let $\delta := \min\{\delta_k, \ k = 1, \dots, N\}$. Since $u(t) \subset \mathcal{U}, \ t \in J(u_0)$, the solution of (5.6) exists at least on the interval $[0, \delta]$. By uniqueness it holds that $v(s) = u(t + s)$ if $t + s \in J(u_0)$, $t \in J(u_0)$, $s \in [0, \delta]$, hence $\sup J(u_0) = +\infty$, i.e. the solution exists globally.

By continuous dependence on the initial data, the solution operator $G_1 : \mathcal{U} \to \mathbb{E}_{1,\mu}(0,\delta)$, which assigns to each initial value $u_1 \in \mathcal{U}$ a unique solution $v(\cdot, u_1) \in \mathbb{E}_{1,\mu}(0,\delta)$, is continuous. Furthermore

$$(\delta/2)^{1-\mu}\|v\|_{\mathbb{E}_1(\delta/2,\delta)} \leq \|v\|_{\mathbb{E}_{1,\mu}(\delta/2,\delta)} \leq \|v\|_{\mathbb{E}_{1,\mu}(0,\delta)},\ \mu \in (1/p, 1),$$

wherefore the mapping $G_2 : \mathbb{E}_{1,\mu}(0,\delta) \to \mathbb{E}_1(\delta/2,\delta)$ with $v \mapsto v$ is continuous. Finally

$$\|v(\delta)\|_{X_\gamma} \leq \|v\|_{BUC((\delta/2,\delta);X_\gamma)} \leq C(\delta)\|v\|_{\mathbb{E}_1(\delta/2,\delta)},$$

hence the mapping $G_3 : \mathbb{E}_1(\delta/2,\delta) \to X_\gamma$ with $v \mapsto v(\delta)$ is continuous. This yields the continuity of the composition $G = G_3 \circ G_2 \circ G_1 : \mathcal{U} \to X_\gamma$, whence $G(\{u(t)\}_{t\geq 0}) = \{u(t+\delta)\}_{t\geq 0}$ is relatively compact in X_γ, since the continuous image of a relatively compact set is relatively compact. Since the solution has relatively compact range in X_γ, it is an easy consequence that the ω-limit set

$$\omega(u_0) := \{v \in V_\mu \cap X_\gamma :\ \exists\, t_n \nearrow \infty \text{ s.t. } u(t_n; u_0) \to v \text{ in } X_\gamma\}$$

is nonempty, connected and compact. We summarize the preceding considerations in the following

Theorem 5.2.1 ([14]). *Let $p \in (1,\infty)$ and suppose that $A(v) \in \mathcal{MR}_p(X_0)$ for all $v \in V_\mu$ and let* (5.2) *as well as* (5.5) *hold for some $\mu \in (1/p, 1)$. Assume furthermore that the solution $u(t)$ of* (5.1) *satisfies*

$$u \in BC([\tau, t^+(u_0)); V_\mu \cap X_\gamma)$$

for some $\tau \in (0, t^+(u_0))$ and

$$\operatorname{dist}(u(t), \partial V_\mu) \geq \eta > 0$$

for all $t \in J(u_0)$. Then the solution exists globally and for each $\delta > 0$, the orbit $\{u(t)\}_{t\geq\delta}$ is relatively compact in X_γ. If in addition $u_0 \in V_\mu \cap X_\gamma$, then $\{u(t)\}_{t\geq 0}$ is relatively compact in X_γ.

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